Coupled non-equilibrium growth equations: Self consistent mode coupling using vertex renormalization

[1] Amit Kr. Chattopadhyay,[2] Abhik Basu,
[1] Jayanta K. Bhattacharjee,

[1] Dept. of Theoretical Physics,
Indian Association for the Cultivation of Science,
Jadavpur, Calcutta 700 032

[2] Dept. of Physics,
Indian Institute of Science, Bangalore 560 012

February 2, 2008

Abstract

We find that studying the simplest of the coupled non-equilibrium growth equations of Barabasi by self-consistent mode coupling requires the use of dressed vertices. Using the vertex renormalization, we find a roughening exponent which already in the leading order is quite close to the numerical value.

PACS number(s):05.10.Gg,05.40.-a,05.70.-a,64.60.Ht

Models of interfacial growth have attracted a tremendous amount of attention since the pioneering work of Kardar, Parisi, Zhang (KPZ) [1,2]. A variety of interesting issues are associated with the KPZ equation and they have given rise to a variety of novel techniques [3]. Among the first analytic techniques used to tackle the KPZ system were the dynamic renormalization group (DRG) [4] and the self-consistent mode coupling scheme (SCMC) [5,6]. An important variant of the KPZ system was introduced by Ertaas and Kardar [7] and Barabasi [8]. This variant consisted of two coupled fields (as opposed to one field in KPZ) and is useful for studying the effects of a second non-equilibrium field on the growing interface. In these coupled field problems DRG has been employed, as also numerical techniques. One does not always get a stable fixed point with the DRG analysis which may sometimes indicate a failure of the perturbation scheme or may indicate a basic instability of the system. It is interesting to note that in many cases the exponents coming from the one-loop DRG analysis are not in very good agreement with the numerical analysis. This is exemplified in the simplest situation treated by Barabasi - an essentially linear system coupled according to

$$\frac{\partial \phi}{\partial t} = \Gamma_1 \frac{\partial^2 \phi}{\partial x^2} + N_1 \tag{1}$$

$$\frac{\partial \psi}{\partial t} = \Gamma_2 \frac{\partial^2 \psi}{\partial x^2} + \lambda \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} + N_2 \tag{2}$$

with $< N_{1,2}(x_1, t_1)\, N_{1,2}(x_2, t_2) >= 2D_{1,2}\, \delta(x_1 - x_2)\, \delta(t_1 - t_2)$.

The field ϕ satisfies the Edwards-Wilkinson equation and the field ψ is coupled linearly via a gradient coupling to the ϕ-field. While the Edwards-Wilkinson model can be exactly solved, this is not true for eqn.(2) because of the multiplicative noise (note that ϕ is a random field). The DRG recursion relations in this case yield for the roughening exponent α of the ψ-field, the value $\alpha = 5/6$ while the numerical value of α is nearly 0.68. The dynamical exponent z of the ψ-field is found to be 2. Thus, in this case the dynamic exponent for both ϕ and ψ fields is found to be 2. We will call this "extended" dynamic scaling i.e. the time scale is independent of the nature of the field [9]. As it turns out, this is the only situation for this case. However this need not always be so. In another model considered by Ertaas and Kardar and Barabasi,

$$\frac{\partial \phi}{\partial t} = \Gamma_1 \frac{\partial^2 \phi}{\partial x^2} + \lambda_1 (\frac{\partial \phi}{\partial x})^2 + N_1 \tag{3}$$

$$\frac{\partial \psi}{\partial t} = \Gamma_2 \frac{\partial^2 \psi}{\partial x^2} + \lambda \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} + N_2 \tag{4}$$

there are two possibilities:
i)$z_\phi = z_\psi = 3/2$, this is the extended dynamic scaling and is found to be the correct situation for $\lambda > 0$ with $\lambda_1 > 0$,
ii) $z_\phi = 3/2$, but $z_\psi = 2$, this situation is obtained for $\lambda < 0$ with $\lambda_1 > 0$ and can be described as "weak" scaling [9]. For problems involving two or more coupled fields, one needs to differentiate between "extended" and "weak" scaling.

In the one-dimensional KPZ, the perturbative DRG is exact (due to the existence of a fluctuation-dissipation relation), but this is not true for the coupled system in one dimension.The self-consistent mode coupling (SCMC) which has been reasonably succesful for the KPZ, has never been attempted in the coupled system. In this note, we apply the SCMC to the coupled system to see if it is a quantitatively better scheme than the perturbative DRG. In the process, we find something quite unusual. In all known situations, SCMC has been succesful in cases where the vertex is not renormalized. This, in contrast, is a situation where the momentum dependence of the dressed vertex is absolutely essential. This is what makes the application of SCMC interesting in this problem and should act as a prototype for situations where dressed vertices are unavoidable. Writing eqns. (1) and (2) in momentum space, we have

$$\dot{\phi(k)} = -\Gamma_1 \, k^2 \, \phi(k) \; + \; N_1(k) \tag{5}$$

$$\dot{\psi(k)} = -\Gamma_2 \, k^2 \, \psi(k) \; - \; \lambda \sum_p p(k-p) \, \phi(p)\psi(k-p) \; + \; N_2(k) \tag{6}$$

with $< N_{1,2}(k_1, \omega_1) N_{1,2}(k_2, \omega_2) > = \; 2D_{1,2} \, \delta(k_1 + k_2) \, \delta(\omega_1 + \omega_2)$.i

The basic elements of the calculation are the Green's functions $G_\phi(k,\omega)$ and $G_\psi(k,\omega)$, the correlation functions $C_\phi(k,\omega)$ and $C_\psi(k,\omega)$ and the vertex function $\Lambda(k, q, k-q)$. Obviously G_ϕ and C_ϕ are exactly known and are given by

$$G_\phi^{-1}(k,\omega) = -i\omega \; + \; \Gamma_1 k^2 \tag{7}$$

$$C_\phi(k,\omega) = \frac{2D_1}{\omega^2 + {\Gamma_1}^2 k^4} \tag{8}$$

while for the ψ-field

$$G_\psi^{-1}(k,\omega) = -i\omega \; + \; \Gamma_2 k^2 \; + \; \Sigma(k,\omega) \tag{9}$$

$$C_\psi(k,\omega) = \frac{2D_2}{\omega^2 + {\Gamma_2}^2 k^4} \; + \mid G_\psi(k,\omega) \mid^2 F(k,\omega) \tag{10}$$

and

$$\Lambda(k,p,k-p) = \lambda \, + \, \Lambda(k,p) \tag{11}$$

The self-energy $\Sigma(k,\omega)$ is found at the dressed one level to be given by

$$\begin{aligned}\Sigma(k,\omega) &= \lambda^2 \int \frac{dp}{2\pi}\frac{d\omega'}{2\pi}\, kp^2(k-p)\, C_\phi(p,\omega')G_\psi(k-p,\omega-\omega') \\ &= \frac{\lambda^2 D_1}{\Gamma} \int \frac{dp}{2\pi} \frac{k(k-p)}{-i\omega + \Gamma_1 p^2 + \Sigma(k-p) + \Gamma_2(k-p)^2}\end{aligned} \tag{12}$$

where we have used eq.(8) and eq.(9) in the Lorentzian approximation, i.e. during the frequency convolution, $\Sigma_\psi(k,\omega)$ has been replaced by its zero frequency form.

Our first observation is that within the extended dynamic scaling, we expect $z_\psi = 2$. We need to examine if this is self-consistent. Setting $\Sigma(k) = \Gamma k^2$, we have

$$\Gamma k^2 = \frac{\lambda^2 D_1}{\Gamma_1} \int \frac{dp}{2\pi} \frac{k(k-p)}{\Gamma_1 p^2 + \tilde{\Gamma_2}(k-p)^2} \tag{13}$$

where $\tilde{\Gamma_2} = \Gamma_2 + \Gamma$.

The long wave length property ($k \to 0$) of the integral on the right hand side is best seen by changing to the symmetric variables $p' = \frac{-k}{2} + p$ which gives the o(k^2) contribution of the integral to be $k^2 \, \frac{\lambda^2 D_1}{\Gamma_1} \, \int \, \frac{dp'}{2\pi} \, \frac{3\Gamma_1 - \tilde{\Gamma_2}}{(\Gamma_1+\Gamma_2)} \, \frac{1}{p'^2}$. This integral is divergent and needs to be cut-off at o(k), which spoils the k^2 behaviour. The only way this can be prevented is by setting $3\Gamma_1 = \tilde{\Gamma_2}$, which makes the o($k^2$) contribution of Σ vanish, i.e. implies $\Gamma = 0$ and this establishes

$$3\Gamma_1 = \Gamma_2 \tag{14}$$

which is in exact agreement with the earlier work of Barabasi.

We now discuss the correlation function. The diagram with bare vertex is shown in Fig.1a and leads to

$$\begin{aligned}C_\psi(k,\omega) &= \frac{2D_2}{\omega^2 + {\Gamma_2}^2 k^4} + \mid G_\psi(k,\omega) \mid^2 \lambda^2 \int \frac{dp}{2\pi}\frac{d\omega'}{2\pi} \\ &\quad p^2(k-p)^2\, C_\phi(p,\omega')C_\psi(k-p,\omega-\omega')\end{aligned} \tag{15}$$

We now assume the scaling form

$$C_\psi(k,\omega) = \frac{D_\psi}{k^{3+2\alpha}}\, f(\omega/k^2) \tag{16}$$

which is consistent with the equal time correlation function, $\int \frac{d\omega}{2\pi} C_\psi(k,\omega)$ being $k^{-1-2\alpha}$. In the absence of λ, $\alpha = 1/2$ and the extra roughness produced by this added noise is expected to raise α beyond $1/2$. Our expectation,

then is that the second term will dominate in eqn.(15). The power count of the second term in eqn.(15) shows that $C_\psi(k,\omega) \sim k^{-4-2\alpha}$ which cannot match the power count of the left hand side for any value of α and hence a self-consistent formulation requires the vertex to acquire a momentum dependence. Dressing the vertex leads to the diagram in Fig.1b. Dropping the first term on the right hand side of eqn.(15) and dressing the vertex in the second leads to

$$\begin{aligned} C_\psi(k,\omega) &= \mid G_\psi(k,\omega)\mid^2 \lambda \int \frac{dp}{2\pi}\frac{d\omega'}{2\pi} p^2 (k-p)^2\, \Lambda(k,p,k-p) \\ & \quad C_\phi(p,\omega')C_\psi(k-p,\omega-\omega') \end{aligned} \tag{17}$$

Since, we are interested in the $k \to 0$ property of $C_\psi(k,\omega)$, the vertex that we need is $Lim_{k\to 0}\Lambda(k,p,k-p)$ and if in this limit the vertex has the form $\Lambda_0 p$ where Λ_0 is a constant, then the self-consistency in power counting is restored. The consistency of the amplitude is assured if (we evaluate the integral in eqn.(17) in the leading approximation [12] of $k \to 0$)

$$\begin{aligned} 1 &= \frac{\lambda D_1}{9{\Gamma_1}^2}\int \frac{dp}{2\pi}\frac{(k-p)^2}{\mid k-p\mid^{1+2\alpha}}\frac{\Lambda(k,p,k-p)}{(k-p)^2+\frac{1}{3}p^2} \\ &\simeq \frac{\lambda\Lambda_0 D_1}{9{\Gamma_1}^2}\int \frac{dp}{2\pi}\frac{1}{\frac{4}{3}p^{2\alpha}} \\ &= \frac{\lambda\Lambda_0 D_1}{12{\Gamma_1}^2}\frac{1}{2\alpha-1} \end{aligned} \tag{18}$$

We note in passing that the above momentum dependence of the vertex does not alter the conditions of eqn.(14). The self consistent equation for the vertex is shown in Fig.2. Clearly

$$\begin{aligned} \Lambda(k,q,k-q) &= \lambda \int \frac{dp}{2\pi}\frac{d\omega}{2\pi}\, p(p-q)(k-p)^2\, G_\psi(p,\omega)G_\psi(p-q,\omega) \\ & \quad C_\phi(k-p,\omega)\Lambda(p,p-q,q)\Lambda(k-p,p-q,k-q) \end{aligned} \tag{19}$$

Once again, the dressed vertex Λ that we are interested in corresponds to $k \to 0$. This vertex scales as q on the left hand side. Power count of the right hand side shows that it is a linear function of momentum as well and thus the two sides are matched in exponents. To impose the amplitude inconsistency, we evaluate the integral on the right hand side in the dominant region which corresponds to small values of p. This leads to

$$1 = \frac{\lambda\Lambda_0 D_1}{2\pi{\Gamma_1}^2}\frac{2\pi}{4\sqrt{3}} \tag{20}$$

Comparing with eqn.(13), we find

$$\alpha = \frac{1}{2} + \frac{1}{2\sqrt{3}\pi} \simeq 0.59 \tag{21}$$

This is to be compared with the numerical value of $\alpha = 0.68$. For a more careful analysis, eqns.(17) and (19) have to be solved numerically. This is an extremely formidable task because the dependence of Λ on the three variables (two independent) has to be charted out.

As a final point, one would like to show that in this particular case, the weak scaling situation does not arise. If z_ψ were to be different from 2, then for $\Sigma(k,\omega)$ to be at all relevant, z_ψ has to be smaller than 2.This means eqn.(12), would at zero frequency become (we now include the vertex correction)

$$\Sigma(k) = \frac{\lambda^2 D_1}{\Gamma_1} \int \frac{dp}{2\pi} \frac{\Lambda(k,p)k(k-p)}{\Sigma(k-p)} \tag{22}$$

Simple power counting shows that with $\Lambda \propto p$, $z_\psi = 2$, which contradicts our starting assumption that $z_\psi < 2$ and hence there is no self-consistent solution of the weak scaling variety.

We have checked to ensure that for the extended scaling case, the self-consistent scheme does give the roughening exponent. Whether, the scheme can be made to work for the weak scaling situation is under consideration.

1 Acknowledgment

The authors AKC and AB sincerely acknowledge partial financial support fom C. S. I. R., India.

References

[1] M. Kardar, G. Parisi and Y. C. Zhang, Phys. Rev. Letts. **56**, 889 (1996).

[2] For a review see T. Halpin-Healy and Y. C. Zhang, Phys. Reps. **254**, 215 (1995); E. Stanley and A.L.Barabasi, "*FractalConceptsInSurfaceGrowth*", Cambridge Univ. Press, 1996.

[3] For a recent discussion of various techniques, see H. K. Janssen, U. C. Täuber and E. Frey, Euro. Phys. Journal B **9**, 491 (1999).

[4] D. Forster, D. Nelson and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).

[5] J. P. Bouchaud and M. E. Cates, Phys. Rev. E **47**, R1455 (1993).

[6] J. P. Doherty, M. A. Moore, A. J. Bray and J. M. Kim, Phys. Rev. Letts. **72**, 2041 (1994).

[7] D. Ertaas and M. Kardar, Phys. Rev. Letts. **69**, 929 (1992).

[8] A. L. Barabasi, Phys. Rev. A **46**, R2977 (1992).

[9] This is a terminology introduced by De Dominicis and Peliti in a similar context in dynamic critical phenomena, see C. De Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978).

[10] J. K. Bhattacharjee, Jour. Phys. A **31**, L93 (1998).

2 Figure Captions

Fig.1a The self-consistent equation for the correlator with *bare* vertex. The double thick line is the dressed correlator and the double straight line the propagator. The cross stands for the noise.
Fig.1b The self-consistent equation for the correlator with *dressed* vertex. The double thick line is the dressed correlator and the double straight line the dressed propagator. The cross stands for the noise.
Fig.2 The self-consistent equation for the vertex.